Complex Analysis: Final Exam

Aletta Jacobshal 02, Wednesday 1 February 2017, 18:30–21:30
Exam duration: 3 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** at the top of the first page of your exam sheet and on the envelope. **Do NOT seal the envelope!**
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you must explicitly state and verify the corresponding conditions.
- 10 points are "free". There are 6 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (15 points)

Evaluate

$$\text{pv}\int_{-\infty}^{\infty} \frac{1}{(x-1)(x^2+1)}dx$$

using the calculus of residues.

Question 2 (15 points)

Show that if f is analytic in the closed disk $|z| \leq 2$ and if $|f(z)| < 1$ on the circle $|z| = 1$, then the equation $f(z) = z^n$ has exactly n solutions (counting multiplicity) in the open disk $|z| < 1$.
Hint: Rouché's theorem; the conditions for applying the theorem must be explicitly stated and verified.

Question 3 (15 points)

Represent the function

$$f(z) = \frac{z+1}{z-1},$$

(a) (8 points) as a Taylor series around 0 and find its radius of convergence;

(b) (7 points) as a Laurent series in the domain $|z| > 1$.

Question 4 (15 points)

At which points is the function

$$f(z) = x^2 + y^2 + 2ixy,$$

differentiable? Compute the derivative of $f(z)$ at these points.

Question 5 (15 points)

Consider the function

$$f(z) = \frac{\sin z}{z^2}.$$

(a) (6 points) Determine the singularities of $f(z)$ and their type (removable, pole, essential; if pole, specify the order).

(b) (9 points) Show that $f(z)$ does not have an antiderivative in $\mathbb{C} \setminus \{0\}$. *Hint: Compute the integral of f along the unit circle.*

Question 6 (15 points)

(a) (8 points) Prove that

$$\cos z = \cos x \cosh y - i \sin x \sinh y,$$

where $z = x + iy$.

(b) (7 points) Prove that the function

$$u(x, y) = \cos x \cosh y,$$

is harmonic in $\mathbb{R}^2$ and find a harmonic conjugate of $u(x, y)$.